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Bank-to-Turn Optimal Guidance with Linear Exponential Quadratic Gaussian Performance Criterion

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This paper presents an optimal guidance law for controlling the pitch acceleration and roll rate of a bank-to-turn missile with linearized plant dynamics and zero autopilot lag. The closed-form solution of the guidance law is derived by using the linear exponential quadratic Gaussian performance criterion and the corresponding Hamilton–Jacobi–Bellman equation. In addition, a constant acceleration bias is also introduced to prevent excessive roll rate command due to noise. It is shown that the optimal control gain can explicitly take both system and measurement noise covariances into consideration; therefore, the average, maximum, and standard deviation of miss distance can be reduced. Comparisons with those optimal guidance laws developed by the traditional linear quadratic Gaussian method are also made, which prove that the proposed guidance law is always better.

I. Introduction

THE method of bank-to-turn (BTT) guidance is characterized by the missile rolling to make the predict intercept point lying in the maximal acceleration plane of the missile. Therefore, the aerodynamic cross coupling due to the asymmetric shielding effect of skid-to-turn (STT) guidance can be reduced. In the past,[1,2] the optimal guidance laws of a BTT missile were derived by linearizing the nonlinear plant equation and applying optimal control theory to solve the deterministic linear quadratic problem (LQP). The LQP is a simple case of the linear quadratic Gaussian (LQG) problem. The LQG problem[3] of optimal stochastic control is subject to a stochastic linear system with Gaussian system and measurement noises. Its performance criterion is the expected value of a quadratic form of the state and control variables. The LQG problem has two important properties. The first is the separation theorem; i.e., the state estimator and the controller can be designed separately, and the controller is a linear function of the estimated states. The second is the certainty equivalent principle; i.e., the linear controller can be developed by neglecting the system and measurement noises and solving the deterministic LQP directly. Therefore, the control gain is independent of the covariances of system and measurement noises. However, if the performance criterion is chosen to be the expected value of an exponential function with quadratic form, i.e., the linear exponential quadratic Gaussian (LEQG) problem,[4,5] then the optimal control gain would take both system and measurement noise covariances into consideration, which is different from that obtained by the deterministic linear exponential quadratic problem (LEQP), so that the certainty equivalent principle cannot be held.

The aforementioned LEQG method was used to derive the terminal guidance law of an STT missile.[6] The results of simulation indicated that the guidance law obtained by using the LEQG method was more effective in reducing the maximum value and/or variance of miss distance than that obtained by the LQG method for larger system and/or measurement noises. In this paper, the authors apply the LEQG method and derive the corresponding Hamilton–Jacobi–Bellman (HJB) equation to develop a terminal guidance law for controlling the pitch acceleration and roll rate commands of a BTT missile. To the authors' knowledge, this derivation method and the new guidance law have not been published in the past. To simplify the guidance law development, the nonlinear dynamic equation of the BTT missile is linearized and the pitch and roll autopilots are assumed to have zero lag. Since both system and measurement noises cannot be neglected in this case, in Sec. II the HJB equation of the LEQG problem, based on the estimated state, is derived and solved to obtain a closed-form solution. In both Sec. III and the Appendix it is proved that both the pitch acceleration and roll rate commands are explicit functions of the system and measurement noise covariances. This allows the missile to decrease the average, maximum, and standard deviation of miss distance in noisy environments, which is consistent with the aforementioned results[6] of an STT missile. Comparisons with those optimal guidance laws derived by the traditional LQG method with or without constant acceleration bias are also made in Sec. IV. From the results of some practical simulations, it can be seen that for larger system and measurement noise covariances, the results of the system by using the proposed guidance law are always better than those with the LQG methods.

II. General Solution of LEQG Problem

Let the system be represented by the state equation

$$\dot{x} = Fx + Gu + \Gamma w \tag{1}$$

and the measurement process

$$z = Hx + v \tag{2}$$

where x is an n-dimensional state vector; u is an m-dimensional control vector; z is a q-dimensional measurement vector; F, G, Γ, and H are $n\times n$, $n\times m$, $n\times p$, and $q\times n$ matrices, respectively; and w and v are p- and q-dimensional uncorrelated Gaussian white-noise processes with zero mean and covariances

$$\begin{aligned} E[w(t)w^T(\tau)] &= Q(t)\delta(t-\tau) \\ E[v(t)v^T(\tau)] &= R(t)\delta(t-\tau) \end{aligned} \tag{3}$$

respectively. To simplify the notation, the time dependence of the variables is suppressed thereafter.

To simplify the problem and obtain a closed-form solution, the problem is to minimize the following LEQG performance criterion with control weighting only:

$$J(x,t_0) = \sigma E\left\{\exp\left[\frac{\sigma}{2}x^T(t_f)S_f x(t_f) + \frac{\sigma}{2}\int_{t_0}^{t_f}(u-u_b)^T B(u-u_b)\,\mathrm{d}t\right]\right\} \tag{4}$$

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where $E(\cdot)$ and $\exp(\cdot)$ are expectation and exponent function operators, respectively; u_b is the constant input bias; S_f is $n \times n$ positive-semidefinite weighting matrix for the terminal states; B is an $m \times m$ positive-definite weighting matrix for control input; and σ is a real number that is also a weighting factor of the LEQG method. In this paper, we derive an optimal terminal guidance law for a BTT missile with LEQG performance criterion. The proposed guidance law is to minimize the terminal mean-square miss distance as well as minimizing the control input.[1–3,6] This problem is referred to as the terminal-cost problem.[5]

Since the performance index defined by Eq. (4) is a linear quadratic function, the optimal control is a linear combination of states. Thus, the separation theorem as well as a Kalman-like filter[5–7] can be applied to produce the optimal estimated state from the noisy measurements, and the state estimation equation can be written as

$$\dot{\hat{x}} = F\hat{x} + Gu + K(z - H\hat{x}) \tag{5}$$

Since the original input to the optimal controller x should be replaced by $\hat{x}$, the performance criterion of Eq. (4) is reduced to

$$J(\hat{x}, t_0) = \sigma E\left\{\exp\left[\frac{\sigma}{2}\hat{x}^T(t_f)S_f\hat{x}(t_f) + \frac{\sigma}{2}\int_{t_0}^{t_f}(u - u_b)^T B(u - u_b)\,\mathrm{d}t\right]\right\} \tag{6}$$

where the original state vector x is replaced by the estimated state vector $\hat{x}$, and K is the gain of the estimator defined as

$$K = PH^TR^{-1} \tag{7}$$

where the error covariance $P \equiv E[(x - \hat{x})(x - \hat{x})^T]$ is propagated forward in time by the Riccati equation:

$$\dot{P} = FP + PF^T + \Gamma Q \Gamma^T - PH^TR^{-1}HP, \qquad P(t_0) = P_0 \tag{8}$$

Since the estimator is an unbiased estimator, the correction term $z - H\hat{x}$ of the estimator in Eq. (5) may be regarded as an equivalent white noise with zero mean and covariance the same as that of v, i.e.,

$$E[(z - H\hat{x})] = 0 \tag{9a}$$

$$E[(z - H\hat{x})(z - H\hat{x})^T] = R(t)\delta(t - \tau) \tag{9b}$$

Let the value function $V(\hat{x}, t)$ be the minimum performance[8] from t to t_f, i.e.,

$$V(\hat{x}, t) = \min_{u(t)} J(\hat{x}, t) \tag{10}$$

Now the HJB equation, with the reformulated LEQG performance criterion defined by Eqs. (5) and (6), can be derived as[4,6]

$$-V_t(\hat{x}, t) = \min_{u(t)}\left\{\frac{\sigma}{2}(u - u_b)^T B(u - u_b)V(\hat{x}, t) + V_{\hat{x}}^T(\hat{x}, t)(F\hat{x} + Gu) + \frac{1}{2}\mathrm{tr}\left[V_{\hat{x}\hat{x}}(\hat{x}, t)KRK^T\right]\right\} \tag{11}$$

where the subindex denotes the partial derivative with respect to that variable, and $\mathrm{tr}(\cdot)$ is a trace function operator.

Following the optimal control theory,[8] the Hamiltonian function H is related to the derivation of the value function as

$$V_t(\hat{x}, t) = -\min_{u(t)} H(\hat{x}, u, t) \tag{12}$$

Comparing Eqs. (11) and (12) one can find

$$H(\hat{x}, u, t) = \frac{\sigma}{2}(u - u_b)^T B(u - u_b)V(\hat{x}, t) + V_{\hat{x}}^T(\hat{x}, t)(F\hat{x} + Gu) + \frac{1}{2}\mathrm{tr}\left[V_{\hat{x}\hat{x}}(\hat{x}, t)KRK^T\right] \tag{13}$$

Applying the optimal control theorem, the optimal control must satisfy the equation

$$0 = \left.\frac{\partial H}{\partial u}\right|_{u=u^*} = \sigma B(u^* - u_b)V(\hat{x}, t) + G^T V_{\hat{x}}(\hat{x}, t) \tag{14}$$

So from Eq. (14), the optimal control $u^*(t)$ can be obtained as

$$u^*(t) = u_b - (1/\sigma)B^{-1}G^T V_{\hat{x}}(\hat{x}, t)V^{-1}(\hat{x}, t) \tag{15}$$

Suppose the optimal value function $V(\hat{x}, t)$ to be of the form[4,6]

$$V(\hat{x}, t) = \sigma D \exp\left(\frac{\sigma}{2}\hat{x}^T S\hat{x}\right) \tag{16}$$

where D is a scalar function and S is a positive-definite symmetric matrix.

Substituting the corresponding terms defined by Eqs. (15) and (16) into Eq. (11) and equating the corresponding terms, one can obtain the relationships

$$-\dot{D} = \frac{\sigma}{2}D\,\mathrm{tr}(SKRK^T) \tag{17}$$

$$-\dot{S}\hat{x} = SF\hat{x} + F^TS\hat{x} - S(GB^{-1}G^T - \sigma KRK^T)S\hat{x} + SGu_b \tag{18}$$

with the boundary conditions of D and S to be derived from Eqs. (6) and (16), i.e.,

$$D(t_f) = 1, \qquad S(t_f) = S_f \tag{19}$$

Substituting Eq. (16) into Eq. (15), optimal control can be obtained as

$$u^* = u_b - B^{-1}G^TS\hat{x} \tag{20}$$

where S must satisfy the Riccati equation defined by Eq. (18).

A guide for the weighting factor σ of the LEQG problem is as follows: Let the value in parentheses in Eq. (18) be defined as

$$E = GB^{-1}G^T - \sigma KRK^T \tag{21}$$

If G^TG and E are nonsingular, then one has an effective control weighting B_{eff} for the LEQG problem defined as

$$B_{\mathrm{eff}} = G^TE^{-1}G \tag{22}$$

Therefore, by Eq. (21) one can see that there is a positive upper limit of σ, say $\sigma_{\max}$, such that

$$B_{\mathrm{eff}} > 0 \qquad \text{for} \qquad \sigma < \sigma_{\max} \tag{23}$$

If $\sigma \geq \sigma_{\max}$, then E defined by Eq. (21) would become negative semidefinite. Thus, the solution of Riccati equation (18) may not exist, and the guaranteed gain margin (-6 dB $<$ GM $< \infty$ dB) and phase margin ($|\mathrm{PM}| > 60$ deg) of the traditional optimal control system cannot be preserved.

It should be noted that by Eqs. (7), (8), (18), and (20), if σ approaches zero, the control gain of the LEQG method would be approximately that of the LQG method. If the weighting factor $\sigma \neq 0$ in Eq. (18), then the optimal control gain would take both system and measurement noise covariances (Q and R) into consideration, which is different from that obtained by the LQG method. For the case $\sigma < 0$, if the noise intensity R increases, then by Eqs. (18) and (20) both S and the control gain will be decreased. Thus, it means that the response time of the missile will be increased; i.e., the terminal miss distance will be increased. Therefore, this case is not useful for the guidance of missiles.

On the other hand, for $0 < \sigma < \sigma_{\max}$, if the noise intensity increases, then by Eqs. (18) and (20) both S and the control gain will be increased. The increased control effort can reduce the terminal miss distance in the presence of large noise. Thus this case is preferable for the guidance of a missile. Moreover, as σ increases, so do S and the control gain. Thus the control gain of the LEQG method is larger than that of the LQG method. But, if σ becomes too large, then the stable solution of Eq. (18) may not exist, and the system may become unstable. Thus σ must be chosen carefully.

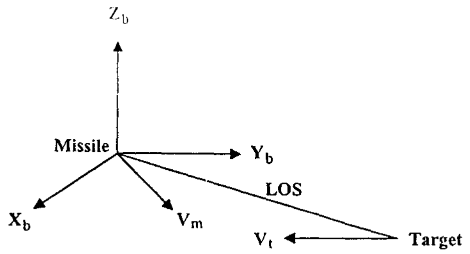

Fig. 1 Relative geometry of missile and target in coordinate of body axis.

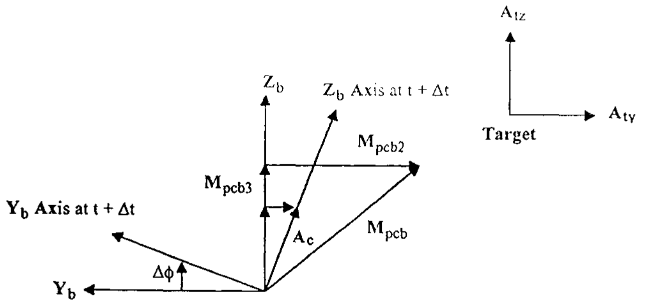

Fig. 2 Incremental roll angle and PCB miss distance in view along missile X_b axis.

III. Bank-to-Turn Optimal Guidance Law Derivation

In this section the linearized plant dynamics of a BTT missile is derived first. Then the optimal guidance law with a constant bias acceleration is derived. The engagement geometry of missile and target is shown in Fig. 1. The miss distance M_{pcb} of missile and target due to the projected constant bias acceleration[2] is shown in Fig. 2, which means that the miss distance is produced for the missile acceleration to be a constant value from the present time to intercept. In addition, M_{pcb2} and M_{pcb3} are the components of M_{pcb} along the Y_b and Z_b axes, respectively. The subindex b denotes the axes to be in the missile body coordinates. Let the target accelerations along the Y_b and Z_b axes be the randomly reversing Poisson square wave,[9] as shown in Fig. 3, and with root-mean-square values β_1 and β_2 (both in meters per square second), respectively. The number of the target's acceleration crossing zero is ν (times per second).

Let the state vector be

$$x(t) = [Y_d, \dot{Y}_d, A_{ty}, Z_d, \dot{Z}_d, A_{tz}, \Delta\phi]^T \tag{24}$$

where Y_d and Z_d are the distances between target and missile in the missile Y_b and Z_b axes, respectively, and A_{ty} and A_{tz} are the target acceleration along the missile Y_b and Z_b axes, respectively. The angle $\Delta\phi$ is the incremental roll angle from the present missile axis to the future orientation for an incremental period Δt.

The control vector is

$$u(t) = [A_c, P_c]^T \tag{25}$$

where A_c is the pitch acceleration command and P_c is the roll rate command.

From Fig. 2 and since target acceleration can be modeled by passing a white noise with power spectral density β^2/ν through a first-order shaping filter with bandwidth 2ν, one can obtain[1,2,9]

$$\frac{d}{dt}\dot{Y}_d = A_{ty} + A_c \sin(\Delta\phi) \tag{26a}$$

$$\frac{d}{dt}\dot{Z}_d = A_{tz} - A_c \cos(\Delta\phi) \tag{26b}$$

$$\frac{d}{dt}A_{ty} = -2\nu A_{ty} + 2\nu w_1 \tag{26c}$$

$$\frac{d}{dt}A_{tz} = -2\nu A_{tz} + 2\nu w_2 \tag{26d}$$

$$\frac{d}{dt}\Delta\phi = P_c \tag{26e}$$

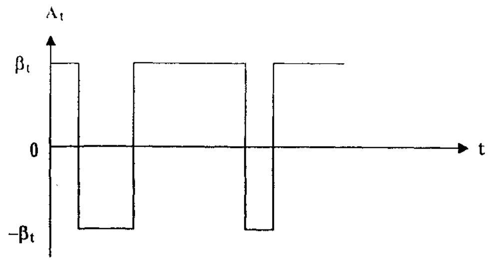

Fig. 3 Poisson square-wave model of target's acceleration.

So the nonlinear plant equation can be obtained as

$$\dot{x} = \begin{bmatrix} \dot{Y}_d \\ A_{ty} + A_c \sin(\Delta\phi) \\ -2\nu A_{ty} \\ \dot{Z}_d \\ A_{tz} - A_c \cos(\Delta\phi) \\ -2\nu A_{tz} \\ P_c \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 2\nu & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 2\nu \\ 0 & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \tag{27}$$

where w_1 and w_2 are Gaussian white noises with zero mean and variances as[9]

$$E[w_1(t)w_1(\tau)] = Q_1\delta(t-\tau) = \beta_1^2/\nu \tag{28a}$$

$$E[w_2(t)w_2(\tau)] = Q_2\delta(t-\tau) = \beta_2^2/\nu \tag{28b}$$

In order to simplify the development, two assumptions are made. First, the incremental period Δt is supposed to be very small, so that $\sin(\Delta\phi)$ and $\cos(\Delta\phi)$ may be approximated by $\Delta\phi$ and 1, respectively. Therefore, Eq. (27) may be rewritten as

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & A'_c \\ 0 & 0 & -2\nu & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2\nu & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} Y_d \\ \dot{Y}_d \\ A_{ty} \\ Z_d \\ \dot{Z}_d \\ A_{tz} \\ \Delta\phi \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ -1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A_c \\ P_c \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 2\nu & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 2\nu \\ 0 & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

$$= Fx + Gu + \Gamma w \tag{29}$$

where A'_c is actually the unknown optimal acceleration command A_c, which can be linearized by the second assumption[2] as follows:

$$A'_c = A_{c0} + (A_b - A_{c0})(t/t_f) \tag{30}$$

where A_{c0} is A_c at $t = 0$ and A_b is the desired constant acceleration bias.

The measurement process is

$$Z = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = Hx + v \tag{31}$$

where v_1 and v_2 are Gaussian white noises with zero mean and variances

$$E[v_1(t)v_1(\tau)] = R_1\delta(t-\tau) \tag{32a}$$

$$E[v_2(t)v_2(\tau)] = R_2\delta(t-\tau) \tag{32b}$$

Let the performance criterion be in the LEQG form with a constant bias acceleration:

$$J = \sigma E\left[\exp\left(\frac{\sigma}{2}x^T(t_f)S_f x(t_f) + \frac{\sigma}{2}\int_{t_0}^{t_f}\left[(u-u_b)^T B(u-u_b)\right]\mathrm{d}t\right)\right] \tag{33}$$

where S_f, u_b, and B are chosen as

$$S_f(i,j) = \begin{cases} 1 & \text{if } i = j = 1,4 \\ 0 & \text{otherwise}\end{cases}$$

$$u_b = [A_b, 0]^T$$

$$B = \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix}$$

The estimator gain can be obtained as

$$K = PH^T R^{-1} = \begin{bmatrix} K_{11} & K_{12} & K_{13} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & K_{21} & K_{22} & K_{23} & 0 \end{bmatrix}^T \tag{34}$$

where

$$K_{11} = 2\left(\frac{4\nu^2 Q_1}{R_1}\right)^{\frac{1}{6}}, \qquad K_{12} = 2\left(\frac{4\nu^2 Q_1}{R_1}\right)^{\frac{1}{3}}$$

$$K_{13} = \left(\frac{4\nu^2 Q_1}{R_1}\right)^{\frac{1}{2}}, \qquad K_{21} = 2\left(\frac{4\nu^2 Q_2}{R_2}\right)^{\frac{1}{6}} \tag{35}$$

$$K_{22} = 2\left(\frac{4\nu^2 Q_2}{R_2}\right)^{\frac{1}{3}}, \qquad K_{23} = \left(\frac{4\nu^2 Q_2}{R_2}\right)^{\frac{1}{2}}$$

Following the procedure mentioned in Sec. II, the pitch acceleration and roll rate commands can be derived as (the details are listed in the Appendix)

$$A_c(t) = A_b + \frac{3(t_f - t)}{3b_1(1-\sigma R_2 E_2) + t_f^3}\hat{M}_{\mathrm{pcb3}} \tag{36}$$

$$P_c(t) = -\frac{2(t_f - t)^3 A_{c0} + \left(2t^3 - 3t^2 t_f + t_f^3\right)A_b}{6t_f\left[b_2(1-\sigma R_1 E_1) + \left(t_f^5/1260\right)\left(13A_b^2 + 30A_b A_{c0} + 20A_{c0}^2\right)\right]} \times \hat{M}_{\mathrm{pcb2}} \tag{37}$$

where E_1, E_2, and $\hat{M}_{\mathrm{pcb2}}$ and $\hat{M}_{\mathrm{pcb3}}$ are defined by Eqs. (A35), (A36), and (A38) in the Appendix. By Eq. (36) one has

$$A_{c0} = A_c(0) = A_b + \frac{3t_f}{3b_1(1-\sigma R_2 E_2) + t_f^3}\hat{M}_{\mathrm{pcb3}} \tag{38}$$

Therefore, by Eqs. (36) and (38), one can prove the assumption of Eq. (30) to be true as follows:

$$\begin{aligned} A_{c0} + (A_b - A_{c0})\frac{t}{t_f} &= A_b + \frac{3t_f}{3b_1(1-\sigma R_2E_2)+t_f^3}\hat{M}_{\mathrm{pcb3}} \\ &\quad + \left(A_b - A_b + \frac{3t_f}{3b_1(1-\sigma R_2E_2)+t_f^3}\hat{M}_{\mathrm{pcb3}}\right)\frac{t}{t_f} \\ &= A_b + \frac{3(t_f - t)}{3b_1(1-\sigma R_2E_2)+t_f^3}\hat{M}_{\mathrm{pcb3}} \\ &= A_c(t) \end{aligned} \tag{39}$$

It should be noted that the pitch acceleration command defined by Eq. (38) can be reformulated to the proportional navigation guidance form as an STT acceleration command obtained by the LEQG method,[6] i.e.,

$$A_c(0) = A_b + \frac{\mathrm{NR}}{t_f^2}\hat{M}_{\mathrm{pcb3}} \tag{40}$$

where NR is the effective navigation ratio defined as

$$\mathrm{NR} = \frac{3t_f^3}{3b_1(1-\sigma R_2 E_2) + t_f^3} \tag{41}$$

Substituting Eq. (A38) into Eq. (40), one can obtain[6]

$$A_c(0) = \mathrm{NR}\frac{\hat{Z}_d(0)}{t_f^2} + \mathrm{NR}\frac{\hat{\dot{Z}}_d(0)}{t_f} + \frac{\mathrm{NR}}{4\nu^2}\left(e^{-2\nu t_f} + 2\nu t_f - 1\right)\hat{A}_{tz}(0) + \left(1 - \frac{\mathrm{NR}}{2}\right)A_b \tag{42}$$

Comparing Eqs. (36) and (37) with the BTT guidance law derived by the LQG method,[2] one can find that the LEQG guidance commands have two additional terms, i.e., $\sigma R_1 E_1$ and $\sigma R_2 E_2$. It should be noted that if σ is zero, then both guidance laws are equivalent. From Eqs. (A33) and (A34) defined in the Appendix, one can determine that both E_1 and E_2 are always positive and R_1 and R_2 are also positive numbers. As discussed in Sec. II, for the case $\sigma < 0$ and both R_1 and R_2 sufficiently large, the commands A_c and P_c are reduced to A_b and 0, respectively. It means that the missile cannot respond to the practical engagement in real time, and the terminal miss distance will be increased.

On the other hand, for the case of $0 < \sigma < \sigma_{\max}$ and both R_1 and R_2 increased, then commands A_c and P_c will also be increased. Although the increased acceleration command can reduce the terminal miss distance, the larger roll rate command P_c will make the missile roll excessively. Nevertheless, a constant acceleration bias can be introduced to prevent excessive roll rate command due to noise.[2]

IV. Simulation Results and Discussion

In this section the guidance law defined by Eqs. (36) and (37) is programmed into digital simulation. The commands A_c and P_c are updated every 0.05 s. For comparison with the previous results obtained by the LQG method,[1] the weighting parameters b_1 and b_2 of the commands A_c and P_c are set to be 0.00578 and 5, respectively. The acceleration bias A_b was chosen to be 0 or 3 g (g is the gravitational acceleration at the surface of Earth, 9.8 m/s^2), which is a reasonable value for the missile. The weighting parameter σ of the LEQG method is chosen to be one of the four values 0.01, 0.001, 0.0001, and 0.00001.

The scenario of target and missile is shown in Fig. 4. Initially, the target and missile are in the same vertical plane, but the target starts to maneuver with acceleration as shown in Fig. 3, and the root-mean-square value is 1 g along the Y_b and Z_b axes for the time to go to be less than 5 s. The rate of the target's acceleration crossing zero is 0.012 times per second.

Three different values of measurement noise covariance σ_v are used in the simulation with 256 runs for each case, where the receiver

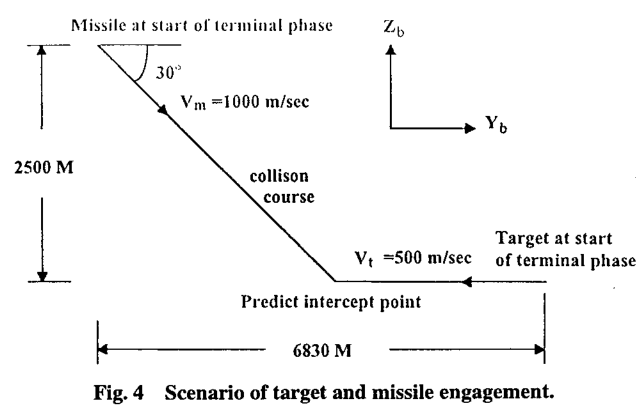

Fig. 4 Scenario of target and missile engagement.

Table 1 Comparisons with guidance laws obtained by LQG and LEQG methods for system with σ_v = 0.01 deg

Acceleration bias A_b	Guidance method	σ Value	Average MD, m	Maximum MD, m	Standard deviation of MD, m	Average total RA, deg	Maximum total RA, deg	Standard deviation of net RA, deg
Unbiased $A_b = 0\,g$	LQG	Not available	5.90	11.93	1.63	214.72	410.26	54.33
	LEQG	0.00001	5.90	11.93	1.63	214.77	410.13	54.33
		0.0001	5.89	11.93	1.63	215.23	408.89	54.36
		0.001	5.86	11.91	1.70	220.75	404.87	55.98
		0.01	5.92	11.57	1.66	240.77	449.81	62.21
Biased $A_b = 3\,g$	LQG	Not available	6.18	9.63	1.14	82.88	164.92	25.26
	LEQG	0.00001	6.18	9.63	1.14	82.88	164.92	25.26
		0.0001	6.18	9.63	1.14	82.88	164.92	25.26
		0.001	6.18	9.63	1.14	82.88	164.89	25.25
		0.01	6.19	9.63	1.14	82.93	164.56	25.16

Table 2 Comparisons with guidance laws obtained by LQG and LEQG methods for system with σ_v = 0.1 deg

Acceleration bias A_b	Guidance method	σ value	Average MD, m	Maximum MD, m	Standard deviation of MD, m	Average total RA, deg	Maximum total RA, deg	Standard deviation of net RA, deg
Unbiased $A_b = 0\,g$	LQG	Not available	7.33	28.88	4.09	382.04	606.79	67.33
	LEQG	0.00001	7.30	28.63	4.11	382.19	607.12	67.39
		0.0001	7.28	21.20	3.92	385.46	609.93	68.50
		0.001	7.41	18.80	3.87	406.84	619.99	66.96
		0.01	7.18	17.27	3.56	474.77	752.14	84.28
Biased $A_b = 3\,g$	LQG	Not available	7.40	17.34	3.78	153.43	284.90	41.74
	LEQG	0.00001	7.40	17.34	3.78	153.44	284.90	41.74
		0.0001	7.40	17.33	3.78	153.46	284.93	41.74
		0.001	7.40	17.28	3.78	153.75	285.18	41.73
		0.01	7.39	17.12	3.78	157.02	287.81	41.69

Table 3 Comparisons with guidance laws obtained by LQG and LEQG methods for system with σ_v = 0.5 deg

Acceleration bias A_b	Guidance method	σ value	Average MD, m	Maximum MD, m	Standard deviation of MD, m	Average total RA, deg	Maximum total RA, deg	Standard deviation of net RA, deg
Unbiased $A_b = 0\,g$	LQG	Not available	17.44	57.53	9.04	495.98	760.90	78.13
	LEQG	0.00001	17.46	57.47	9.06	496.72	761.89	78.43
		0.0001	17.28	56.95	9.00	504.67	765.53	81.97
		0.001	17.83	53.94	8.79	523.69	766.42	86.35
		0.01	17.33	48.30	8.87	603.11	863.40	97.88
Biased $A_b = 3\,g$	LQG	Not available	17.75	55.07	9.55	292.96	542.01	66.28
	LEQG	0.00001	17.75	55.07	9.55	293.03	542.09	66.31
		0.0001	17.74	55.05	9.54	293.73	542.83	66.58
		0.001	17.72	54.88	9.48	301.29	550.48	69.84
		0.01	17.77	51.52	9.51	372.58	689.86	86.60

noise covariance σ_v is obtained by dividing the measurement noise covariance R with the range of missile to target. Some statistical results, e.g., the average, maximum, and standard deviation of miss distance (MD in meters) as well as total rolling angle (RA in degrees, the summation of absolute rolling angle for each updated period) are listed in Tables 1–3. In these tables, the results obtained by the LQG method are also listed for comparison.

From the results of simulation one can see that, when the measurement noise covariance is small ($\sigma_v = 0.01$ deg), the results obtained by using the LEQG method and LQG method are almost identical. But for the larger noise conditions (σ_v of 0.1 or 0.5 deg), one can see that the LEQG method are better than those obtained by the LQG method in miss distance. This is consistent with the results derived in Secs. II and III. This proves that the LEQG problem is equivalent to the min–max optimal control problem of noncooperative differential games.[4,5]

From Tables 2 and 3, it can be seen that the weighting factor σ of the LEQG method is a critical parameter in a noisy environment with $A_b = 0\,g$. Although the miss distance performances with $\sigma = 0.01$ are better in these conditions, the rolling angles are excessive. The reason is that, if $0 < \sigma < \sigma_{\max}$, then by Eqs. (18) and (20) the Riccati solution S as well as the optimal control gain would be respectively larger than those with $\sigma = 0$; i.e., the bandwidth of the system increases. Thus the rolling angle would be sensitive to the noise if the acceleration bias is not included. Three methods can be used to solve this problem. First, one can select another weighting factor b_2, but for comparison with those results reported previously,[1] b_2 is not changed in this paper. Second, one can select a lower weighting value σ; then the excessive rolling angle can be reduced, as shown in Tables 2 and 3. The third method is to add a constant acceleration bias into command A_c. From the results of simulation one can see that the acceleration bias can be used to reduce the total rolling angle, especially for the LEQG method. In summary, when larger measurement and/or system noises are present, the weighting factor σ and constant acceleration bias of the proposed method can be adjusted to yield a better performance.

V. Conclusions

In this paper, a new optimal guidance law for a BTT missile was derived by using the LEQG performance criterion and constant acceleration bias. The advantage of introducing an acceleration bias in the pitch command is that the excessive rolling angle effect of a BTT missile for a noisy environment can be avoided. In addition, it is not necessary to put any weight on the rolling angle state; therefore, a closed-form solution can be obtained. The control gain of the LEQG method can take both system and measurement noise covariances into consideration. By comparison with those results obtained by the LQG method, it was shown that the proposed guidance law is more effective for noisy environments in reducing the miss distance. Some results of the simulation have been given for illustration.

Appendix: Derivation of Optimal Guidance Law

The problem is described by Eqs. (23–34). Following the procedure mentioned in Sec. II, the estimated state equation can be rewritten as

$$\dot{\hat{x}} = F\hat{x} + Gu + K(z - H\hat{x}) \tag{A1}$$

and the performance criterion is

$$J = \sigma E\left[\exp\left(\frac{\sigma}{2}\hat{x}^T(t_f)S_f\hat{x}(t_f) + \frac{\sigma}{2}\int_0^{t_f}\left[(u-u_b)^T B(u-u_b)\right]\mathrm{d}t\right)\right] \tag{A2}$$

where the initial time is set to zero, i.e., $t_0 = 0$.

From Eqs. (19) and (22), one has the optimal control to be as

$$u^*(t) = u_b - B^{-1}G^T S(t)\hat{x}(t) \tag{A3}$$

where $S(t)$ and $\hat{x}(t)$ must satisfy the relationship

$$-\dot{S}\hat{x} = SF\hat{x} + F^T S\hat{x} - S(GB^{-1}G^T - \sigma KRK^T)S\hat{x} + SGu_b \tag{A4}$$

To solve the term $S(t)\hat{x}(t)$ in Eq. (A3), let

$$N(t) = S(t)\hat{x}(t), \qquad N(t_f) = S_f\hat{x}(t_f) \tag{A5}$$

Then

$$\dot{N}(t) = \dot{S}(t)\hat{x}(t) + S(t)\dot{\hat{x}}(t) \tag{A6}$$

Substituting Eqs. (A5) and (A6) into Eq. (A4), one has

$$-\dot{N} + S\dot{\hat{x}} = SF\hat{x} + F^T N - S(GB^{-1}G^T - \sigma KRK^T)N + SGu_b \tag{A7}$$

Since Eq. (A7) must hold for any $S(t)$, we obtain

$$\dot{\hat{x}} = F\hat{x} - (GB^{-1}G^T - \sigma KRK^T)N + Gu_b \tag{A8}$$

$$\dot{N} = -F^T N \tag{A9}$$

Let $\Phi(t,\tau)$ and $\Psi(t,\tau)$ be the state transition matrices of $F(t)$ and $-F^T(t)$, respectively. Then one can obtain

$$\Phi(t,\tau) = \begin{bmatrix} 1 & \phi_{12} & \phi_{13} & 0 & 0 & 0 & \phi_{17} \\ 0 & 1 & \phi_{23} & 0 & 0 & 0 & \phi_{27} \\ 0 & 0 & \phi_{33} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \phi_{45} & \phi_{46} & 0 \\ 0 & 0 & 0 & 0 & 1 & \phi_{56} & 0 \\ 0 & 0 & 0 & 0 & 0 & \phi_{66} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{A10}$$

$$\Psi(t,\tau) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \psi_{21} & 1 & 0 & 0 & 0 & 0 & 0 \\ \psi_{31} & \psi_{32} & \psi_{33} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \psi_{54} & 1 & 0 & 0 \\ 0 & 0 & 0 & \psi_{64} & \psi_{65} & \psi_{66} & 0 \\ \psi_{71} & \psi_{72} & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{A11}$$

where

$$\phi_{12}(t,\tau) = \phi_{45}(t,\tau) = t - \tau \tag{A12}$$

$$\phi_{13}(t,\tau) = \phi_{46}(t,\tau) = \frac{1}{4\nu^2}\left[e^{-2\nu(t-\tau)} + 2\nu(t-\tau) - 1\right] \tag{A13}$$

$$\phi_{23}(t,\tau) = \phi_{56}(t,\tau) = \frac{1}{2\nu}\left[1 - e^{-2\nu(t-\tau)}\right] \tag{A14}$$

$$\phi_{33}(t,\tau) = \phi_{66}(t,\tau) = e^{-2\nu(t-\tau)} \tag{A15}$$

$$\phi_{17}(t,\tau) = \frac{A_{c0}}{2}(t-\tau)^2 + \frac{t^3 - 3t\tau^2 + 2\tau^3}{6t_f}(A_b - A_{c0}) \tag{A16}$$

$$\phi_{27}(t,\tau) = A_{c0}(t-\tau) + \frac{t^2-\tau^2}{2t_f}(A_b - A_{c0}) \tag{A17}$$

$$\psi_{21}(t,\tau) = \psi_{54}(t,\tau) = -(t-\tau) \tag{A18}$$

$$\psi_{31}(t,\tau) = \phi_{64}(t,\tau) = \frac{1}{4\nu^2}\left[e^{2\nu(t-\tau)} - 2\nu(t-\tau) - 1\right] \tag{A19}$$

$$\psi_{32}(t,\tau) = \psi_{65}(t,\tau) = \frac{1}{2\nu}\left[1 - e^{2\nu(t-\tau)}\right] \tag{A20}$$

$$\psi_{33}(t,\tau) = \psi_{66}(t,\tau) = e^{2\nu(t-\tau)} \tag{A21}$$

$$\psi_{71}(t,\tau) = \frac{A_{c0}}{2}(t-\tau)^2 + \frac{2t^3 - 3t^2\tau + \tau^3}{6t_f}(A_b - A_{c0}) \tag{A22}$$

$$\psi_{72}(t,\tau) = -A_{c0}(t-\tau) - \frac{t^2-\tau^2}{-2t_f}(A_b - A_{c0}) \tag{A23}$$

From Eqs. (A12–A23), one can obtain the relationship

$$\Phi(t,\tau) = \Psi^T(\tau,t) \tag{A24}$$

From the definition of the state transition matrix and by Eqs. (A8) and (A9), one can obtain

$$N(t) = \Psi(t,t_f)N(t_f) \tag{A25}$$

and the solution of Eq. (A8) is

$$\hat{x}(t_f) = \Phi(t_f,0)\hat{x}(0) + \int_0^{t_f}\Phi(t_f,\eta)(-GB^{-1}G^T + \sigma KRK^T)N(\eta)\,\mathrm{d}\eta + \int_0^{t_f}\Phi(t_f,\eta)Gu_b\,\mathrm{d}\eta \tag{A26}$$

Postmultiplying both sides of Eq. (A26) by S_f, substituting the corresponding terms of Eqs. (A5) and (A25) into Eq. (A26), and factoring out $S_f\hat{x}(t_f)$, one has

$$S_f\hat{x}(t_f) = \left[I - S_f\int_0^{t_f}\Phi(t_f,\eta)(-GB^{-1}G^T + \sigma KRK^T)\Psi(\eta,t_f)\,\mathrm{d}\eta\right]^{-1} \times S_f\left[\Phi(t_f,0)\hat{x}(0) + \int_0^{t_f}\Phi(t_f,\eta)Gu_b\,\mathrm{d}\eta\right] \tag{A27}$$

Let the value in the inverse term of Eq. (A27) be

$$C = I - S_f \int_0^{t_f} \Phi(t_f, \eta)(-GB^{-1}G^T + \sigma KRK^T)\Psi(\eta, t_f)\,\mathrm{d}\eta$$

$$= \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 & c_{17} \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & c_{45} & c_{46} & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{A28}$$

where

$$c_{11} = 1 - \sigma R_1 E_1 + \frac{t_f^5}{1260 b_2}\left(13A_b^2 + 30A_b A_{c0} + 20A_{c0}^2\right) \tag{A29}$$

$$c_{44} = 1 - \sigma R_2 E_2 + \frac{t_f^3}{3b_1} \tag{A30}$$

From Eqs. (A27) and (A28) it should be noted that only c_{11} and c_{44} are to be computed (the others need not to be found), and the terms E_1 and E_2 in Eqs. (A29) and (A30) are defined as

$$\begin{aligned} E_1 = \int_0^{t_f} \big\{ & K_{11}^2 + \psi_{21}(\eta, t_f)\phi_{12}(t_f, \eta)K_{12}^2 \\ & + \psi_{31}(\eta, t_f)\phi_{13}(t_f, \eta)K_{13}^2 + [\phi_{12}(t_f, \eta) + \psi_{21}(\eta, t_f)]K_{11}K_{12} \\ & + [\phi_{13}(t_f, \eta) + \psi_{31}(\eta, t_f)]K_{11}K_{13} + [\psi_{21}(\eta, t_f)\phi_{13}(t_f, \eta) \\ & + \psi_{31}(\eta, t_f)\phi_{12}(t_f, \eta)]K_{12}K_{13}\big\}\,\mathrm{d}\eta \end{aligned} \tag{A31}$$

$$\begin{aligned} E_2 = \int_0^{t_f} \big\{ & K_{21}^2 + \psi_{54}(\eta, t_f)\phi_{45}(t_f, \eta)K_{22}^2 \\ & + \psi_{64}(\eta, t_f)\phi_{46}(t_f, \eta)K_{23}^2 + [\phi_{45}(t_f, \eta) + \psi_{54}(\eta, t_f)]K_{21}K_{22} \\ & + [\phi_{46}(t_f, \eta) + \psi_{64}(\eta, t_f)]K_{21}K_{23} + [\psi_{54}(\eta, t_f)\phi_{46}(t_f, \eta) \\ & + \psi_{64}(\eta, t_f)\phi_{45}(t_f, \eta)]K_{22}K_{23}\big\}\,\mathrm{d}\eta \end{aligned} \tag{A32}$$

From the relationship defined by Eq. (A24), Eqs. (A31) and (A32) can be rewritten as

$$E_1 = \int_0^{t_f} [K_{11} + \phi_{12}(t_f, \eta)K_{12} + \phi_{13}(t_f, \eta)K_{13}]^2\,\mathrm{d}\eta \tag{A33}$$

$$E_2 = \int_0^{t_f} [K_{21} + \phi_{45}(t_f, \eta)K_{22} + \phi_{46}(t_f, \eta)K_{23}]^2\,\mathrm{d}\eta \tag{A34}$$

From Eqs. (A33) and (A34) it should be noted that the values of both integrals are positive, so that both E_1 and E_2 are always positive. Substituting the corresponding terms of Eqs. (A12–A23) into Eqs. (A33) and (A34), one can obtain

$$\begin{aligned} E_1 = {} & t_f K_{11}^2 + \frac{1}{3}t_f^3 K_{12}^2 + \frac{1}{16\nu^4}\left(-\frac{1}{4\nu}e^{-4\nu t_f} - 2t_f e^{-2\nu t_f} \right. \\ & \left. + \frac{1}{4\nu} + t_f - 2\nu t_f^2 + \frac{4}{3}\nu^2 t_f^3\right)K_{13}^2 + t_f^2 K_{11}K_{12} \\ & + \frac{1}{2\nu^2}\left(-\frac{1}{2\nu}e^{-2\nu t_f} + \frac{1}{2\nu} - t_f + \nu t_f^2\right)K_{11}K_{13} \\ & + \frac{1}{2\nu^2}\left(-\frac{1}{4\nu^2}e^{-2\nu t_f} - \frac{1}{2\nu}t_f e^{-2\nu t_f} + \frac{1}{4\nu^2} - \frac{1}{2}t_f^2 + \frac{2}{3}\nu t_f^3\right) \\ & \times K_{12}K_{13} \end{aligned} \tag{A35}$$

$$\begin{aligned} E_2 = {} & t_f K_{21}^2 + \frac{1}{3}t_f^3 K_{22}^2 + \frac{1}{16\nu^4}\left(-\frac{1}{4\nu}e^{-4\nu t_f} - 2t_f e^{-2\nu t_f} \right. \\ & \left. + \frac{1}{4\nu} + t_f - 2\nu t_f^2 + \frac{4}{3}\nu^2 t_f^3\right)K_{23}^2 + t_f^2 K_{21}K_{22} \\ & + \frac{1}{2\nu^2}\left(-\frac{1}{2\nu}e^{-2\nu t_f} + \frac{1}{2\nu} - t_f + \nu t_f^2\right)K_{21}K_{23} \\ & + \frac{1}{2\nu^2}\left(-\frac{1}{4\nu^2}e^{-2\nu t_f} - \frac{1}{2\nu}t_f e^{-2\nu t_f} + \frac{1}{4\nu^2} - \frac{1}{2}t_f^2 + \frac{2}{3}\nu t_f^3\right) \\ & \times K_{22}K_{23} \end{aligned} \tag{A36}$$

The inverse of C can be obtained as

$$C^{-1} = \begin{bmatrix} \frac{1}{c_{11}} & -\frac{c_{12}}{c_{11}} & -\frac{c_{13}}{c_{11}} & 0 & 0 & 0 & -\frac{c_{17}}{c_{11}} \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{c_{44}} & -\frac{c_{45}}{c_{44}} & -\frac{c_{46}}{c_{44}} & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{A37}$$

By Eqs. (A10–A17) the second term on the right side of Eq. (A27) is

$$\hat{M}_{\mathrm{pcb}} = S_f\left[\Phi(t_f, 0)\hat{x}(0) + \int_0^{t_f} \Phi(t_f, \eta)Gu_b\,\mathrm{d}\eta\right]$$

$$= [\hat{M}_{\mathrm{pcb2}}, 0, 0, \hat{M}_{\mathrm{pcb3}}, 0, 0, 0]^T \tag{A38a}$$

where

$$\hat{M}_{\mathrm{pcb2}} = \hat{Y}_d(0) + t_f\dot{\hat{Y}}_d(0) + \frac{1}{4\nu^2}\left(e^{-2\nu t_f} + 2\nu t_f - 1\right)\hat{A}_{ty}(0) \tag{A38b}$$

$$\begin{aligned} \hat{M}_{\mathrm{pcb3}} = {} & \hat{Z}_d(0) + t_f\dot{\hat{Z}}_d(0) \\ & + \frac{1}{4\nu^2}\left(e^{-2\nu t_f} + 2\nu t_f - 1\right)\hat{A}_{tz}(0) - \frac{t_f^2}{2}A_b \end{aligned} \tag{A38c}$$

The physical meanings of $\hat{M}_{\mathrm{pcb2}}$ and $\hat{M}_{\mathrm{pcb3}}$ in Eq. (A38a) can be seen from Eqs. (A38b) and (A38c), which correspond to the miss distances produced by the projected constant acceleration bias in Y_b and Z_b axes, respectively. Therefore, $\hat{M}_{\mathrm{pcb}}$ is the vector of $\hat{M}_{\mathrm{pcb2}}$ and $\hat{M}_{\mathrm{pcb3}}$ as shown in Fig. 2, where

$$\hat{x}(0) = [\hat{Y}_d(0), \dot{\hat{Y}}_d(0), \hat{A}_{ty}(0), \hat{Z}_d(0), \dot{\hat{Z}}_d(0), \hat{A}_{tz}(0), \Delta\hat{\phi}(0)]^T \tag{A39}$$

and

$$\Delta\hat{\phi}(0) = 0 \tag{A40}$$

Substituting Eqs. (A37) and (A38a) into Eq. (A27), one has

$$S_f\hat{x}(t_f) = C^{-1} \times \hat{M}_{\mathrm{pcb}} = \left[\frac{\hat{M}_{\mathrm{pcb2}}}{c_{11}}, 0, 0, \frac{\hat{M}_{\mathrm{pcb3}}}{c_{44}}, 0, 0, 0\right]^T \tag{A41}$$

By Eqs. (A5) and (A25), one has

$$N(t) = \Psi(t, t_f)N(t_f) = \Psi(t, t_f)S_f\hat{x}(t_f) \tag{A42}$$

Substituting Eqs. (A5), (A41), and (A42) into Eq. (A3), one has the optimal control as

$$u(t) = u_b - B^{-1}G^T S(t)\hat{x}(t)$$

$$= u_b - B^{-1}G^T \Psi(t, t_f)S_f\hat{x}(t_f)$$

$$= u_b - \begin{bmatrix} -\dfrac{\psi_{54}(t, t_f)}{b_1} & \dfrac{\hat{M}_{\mathrm{pcb3}}}{c_{44}} \\ \dfrac{\psi_{71}(t, t_f)}{b_2} & \dfrac{\hat{M}_{\mathrm{pcb2}}}{c_{11}} \end{bmatrix} \tag{A43}$$

Substituting the corresponding terms defined in Eqs. (A18), (A22), (A29), and (A30) into Eq. (A43), one has the optimal guidance law as

$$A_c(t) = A_b + \frac{3(t_f - t)}{3b_1(1 - \sigma R_2 E_2) + t_f^3}\hat{M}_{\mathrm{pcb3}} \tag{A44}$$

$$P_c(t) = -\frac{2(t_f - t)^3 A_{c0} + \left(2t^3 - 3t^2 t_f + t_f^3\right)A_b}{6t_f\left[b_2(1 - \sigma R_1 E_1) + \left(t_f^5/1260\right)\left(13A_b^2 + 30A_b A_{c0} + 20A_{c0}^2\right)\right]} \times \hat{M}_{\mathrm{pcb2}} \tag{A45}$$

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